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Convergence of Padé Approximants for a Certain Class of Meromorphic Functions

N. R. FRANZEN

*Department of Mathematics, Oregon State University, Corvallis, Oregon 97331**Communicated by Oved Shisha*

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In [1] we considered the convergence of diagonal sequences of Padé approximants of meromorphic functions of the special type

$$(1) \quad f(x) = \sum_{i=1}^{\infty} \frac{A_i}{a_i - x},$$

where the A_i and a_i are complex numbers satisfying

$$\sum_{i=1}^{\infty} \left| \frac{A_i}{a_i} \right| < \infty.$$

The method utilized certain series expansions for the persymmetric determinants $\Delta_{m-1,n} = |c_{n+i-j}|_{i,j=0}^{m-1}$, namely,

$$\begin{aligned} \Delta_{m-1,n} &= (-1)^{m(m-1)/2} \sum_{k_1 < k_2 < \dots < k_m} A_{k_1} A_{k_2} \dots A_{k_m} (a_{k_1} a_{k_2} \dots a_{k_m})^{-(n+m)} \\ &\quad \times \prod_{1 \leq i < j \leq m} (a_{k_j} - a_{k_i})^2, \end{aligned}$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

and so

$$c_n = \sum_{i=0}^{\infty} \frac{A_i}{a_i^{n+1}}.$$

We use this expansion below to give an elementary treatment of the convergence of diagonal sequences of Padé approximants of functions of the form

$$(2) \quad f(x) = c_0 + c_1 + \cdots + c_{r-1}x^{r-1} + \sum_{i=1}^{\infty} A_i \left[\frac{1}{a_i - x} - \frac{1}{a_i} - \frac{x}{a_i^2} - \cdots - \frac{x^{r-1}}{a_i^r} \right]$$

satisfying the conditions:

- (i) the c_i ; $i = 0, 1, \dots, r-1$ are arbitrary complex numbers,
- (ii) the A_i are real,
- (iii) the a_i lie on a line L through the origin, i.e., for some θ , $a_i = d_i e^{i\theta}$ for all i with the d_i real, and
- (iv) $0 < |d_1| < |d_2| < |d_3| < \cdots$ with $\sum_{i=1}^{\infty} |A_i/d_i^{r+1}| < \infty$.

When $r = 0$, we take $f(x)$ to be of the form (1). Let S_θ denote the part of the line L given by

$$S_\theta = \begin{cases} \{x : \arg x = \theta, |x| \geq |d_1|\} & \text{if } d_i > 0 \text{ for all } i, \\ \{x : x \in L, |x| \geq |d_1|\} & \text{if } d_i < 0 \text{ for some } i. \end{cases}$$

We let $f_{m,n}(x) = P_{m,n}(x)/Q_{m,n}(x)$ denote the reduced (m, n) -Padé approximant of $f(x)$. We have the following theorem.

THEOREM. *Let $f(x)$ be as described in (2). If p is an integer such that $p \geq r-1$ and $A_i/d_i^p > 0$ for all i , then the sequence $\{f_{m,m+p}(x)\}$ converges to $f(x)$ uniformly on any compact set bounded away from S_θ .*

Remark. When $r = 0$ and $p = -1$ we interpret the sequence of approximants as the diagonal file lying immediately below the principal diagonal file of the Padé table.

The theorem covers three cases.

Case A. If $A_i > 0$ and $d_i > 0$ for all i , we may take $p \geq r-1$ to be arbitrary.

Case B. If $A_i > 0$ for all i , but $d_i < 0$ for some i , we take $p \geq r-1$ to be even.

Case C. If $\text{sign } A_i = \text{sign } d_i$ for all i , we take $p \geq r-1$ to be odd.

To facilitate the exposition of the proof of the theorem we list some notation and formulas which were given in [1] in slightly modified form:

- (3) $\Delta_{m,n}^{(i)}$ and $\Delta_{m,n}(x)$ are the determinants obtained from $\Delta_{m,n}$ by replacing the first row by c_{n+m+i} , $c_{n+m+i-1}$, ..., c_{n+i} and by $1, x, x^2, \dots, x^m$, respectively.

- (4) Let $S(m)$ denote the set of all m -tuples (k_1, k_2, \dots, k_m) of positive integers such that $k_1 < k_2 < \dots < k_m$. For every $k \in S(m)$, define

$$T_{m,m+p}^k = A_{k_1} A_{k_2} \cdots A_{k_m} (d_{k_1} d_{k_2} \cdots d_{k_m})^{-(2m+p)} \prod_{1 \leq i < j \leq m} (d_{k_j} - d_{k_i})^2.$$

- (5) For $p \geq r - 1$,

$$\Delta_{m-1,m+p} = (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T_{m,m+p}^k$$

- (6) If $p \geq r - 1$,

$$\Delta_{m,m+p}(x) = (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T_{m,m+p}^k \prod_{v=1}^m \left(1 - \frac{x}{a_{k_v}}\right).$$

- (7) For $\Delta_{m-1,m+p} \neq 0$,

$$Q_{m,m+p}(x) = \Delta_{m,m+p}(x) / \Delta_{m-1,m+p}.$$

- (8) If $\Delta_{m-1,m+p} \neq 0$,

$$f(x) Q_{m,m+p}(x) - P_{m,m+p}(x) = \Delta_{m-1,m+p}^{-1} \sum_{i=1}^{\infty} \Delta_{m,m+p}^{(i)} x^{2m+p+i}.$$

- (9) If $\Delta_{m-1,n} \neq 0$ and $n \geq m - 1$,

$$P_{m+1,n+1} Q_{m,n} - P_{m,n} Q_{m+1,n+1} = (-1)^m (\Delta_{m,n+1} / \Delta_{m-1,n}) x^{m+n+1}.$$

$$(10) \quad \frac{P_{m,m+p}}{Q_{m,m+p}} = \frac{P_{0,p}}{Q_{0,p}} + \sum_{i=0}^{m-1} (-1)^i \frac{\Delta_{i,i+p+1}}{\Delta_{i-1,i+p}} \frac{x^{2i+p+1}}{Q_{i,i+p} Q_{i+1,i+p+1}}$$

$$\text{if } \Delta_{i-1,i+p} \neq 0 \quad \text{for } i = 0, 1, \dots, m-1.$$

The expansions (5) and (6) were originally given only in the case where $r = 0$ and $\theta = 0$. That they remain valid for functions of the form (2), when $p \geq r - 1$, is easily seen from the fact that the determinants $\Delta_{m-1,m+p}$ and $\Delta_{m,m+p}(x)$ involve only the c_n for $n \geq r$ and we have for $n \geq r$ that

$$c_n = \sum_{i=1}^{\infty} \frac{A_i}{a_i^{n+1}}.$$

The formulas then follow as in [1].

In order to discuss the convergence of the sum (10) we will need appropriate estimates for the quantities $|\Delta_{i,i+p+1} / \Delta_{i-1,i+p}|$ and $|Q_{i,i+p}(x)|$. These are obtained in the following lemmas.

LEMMA 1. Under the conditions of the theorem, $\Delta_{m-1, m+p} \neq 0$ for all m and

$$|\Delta_{m, m+p+1}/\Delta_{m-1, m+p}| \leq |c_{p+1}| |a_1 a_2 \cdots a_m / 2 \cdot 2 \cdots 2|^{-2}.$$

Proof. Observe that because $A_i/d_i^p > 0$ for all i , it follows from (4) that $T_{m, m+p}^k > 0$ for all $k \in S(m)$. Hence

$$\begin{aligned} |\Delta_{m-1, m+p}| &= \left| (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T_{m, m+p}^k \right| \\ &= \sum_{k \in S(m)} T_{m, m+p}^k > 0. \end{aligned}$$

Also

$$\begin{aligned} |\Delta_{m, m+p+1}| &= \sum_{k' \in S(m+1)} T_{m+1, m+p+1}^{k'} \\ &= \sum_{k \in S(m)} \left[T_{m, m+p}^k (d_{k_1} d_{k_2} \cdots d_{k_m})^{-2} \right. \\ &\quad \times \left. \left(\sum_{i=k_m+1}^{\infty} \frac{A_i}{d_i^{p+2}} \right) \left(1 - \frac{d_{k_m}}{d_{k_m+1}} \right)^2 \cdots \left(1 - \frac{d_{k_1}}{d_{k_m+1}} \right)^2 \right] \\ &\leq \left(\sum_{k \in S(m)} T_{m, m+p}^k \right) |a_1 a_2 \cdots a_m|^{-2} \left(\sum_{i=m+1}^{\infty} \frac{A_i}{d_i^{p+2}} \right) 2^{2m} \\ &\leq |\Delta_{m-1, m+p}| |a_1 a_2 \cdots a_m / 2 \cdot 2 \cdots 2|^{-2} |c_{p+1}|. \end{aligned}$$

This proves the lemma.

Let $\tilde{f}(x)$ denote the function obtained from $f(x)$ by setting $\theta = 0$, so that $a_i = d_i$. Let $\tilde{J}_{m-1, m+p}$, $\tilde{J}_{m, m+p}(x)$, $\tilde{P}_{m, m+p}(x)$ and $\tilde{Q}_{m, m+p}(x)$ denote quantities corresponding to $\tilde{f}(x)$. We then have:

LEMMA 2. The polynomials $\tilde{Q}_{m, m+p}(x)$ have only real zeroes.

Proof. From the identity

$$\begin{aligned} \tilde{P}_{m+2, m+p+2} \tilde{Q}_{m, m+p} - \tilde{P}_{m, m+p} \tilde{Q}_{m+2, m+p+2} \\ \equiv [\tilde{f}(x) \tilde{Q}_{m, m+p} - \tilde{P}_{m, m+p}] \tilde{Q}_{m+2, m+p+2} \\ - [\tilde{f}(x) \tilde{Q}_{m+2, m+p+2} - \tilde{P}_{m+2, m+p+2}] \tilde{Q}_{m, m+p} \end{aligned}$$

and (8), we see that the right-hand side is divisible by x^{2m+p+1} . Hence since the left-hand side is a polynomial of degree at most $2m+p+2$ it must have the form

$$(11) \quad (r'_{m, m+p} + s'_{m, m+p} x) x^{2m+p+1}.$$

From (9) we obtain

$$(12) \quad \begin{aligned} & \tilde{P}_{m+1, m+p+1} \tilde{Q}_{m, m+p} - \tilde{P}_{m, m+p} \tilde{Q}_{m+1, m+p+1} \\ &= (-1)^m (\tilde{\Delta}_{m, m+p+1} / \tilde{\Delta}_{m-1, m+p}) x^{2m+p+1}, \end{aligned}$$

and

$$\begin{aligned} & \tilde{P}_{m+2, m+p+2} \tilde{Q}_{m+1, m+p+1} - \tilde{P}_{m+1, m+p+1} \tilde{Q}_{m+2, m+p+2} \\ &= (-1)^{m+1} (\tilde{\Delta}_{m+1, m+p+1} / \tilde{\Delta}_{m, m+p}) x^{2m+p+3}. \end{aligned}$$

Substituting (11) and (12) into the identity

$$\begin{aligned} & [\tilde{P}_{m+1, m+p+1} \tilde{Q}_{m, m+p} - \tilde{P}_{m, m+p} \tilde{Q}_{m+1, m+p+1}] \tilde{Q}_{m+2, m+p+2} \\ &+ [\tilde{P}_{m, m+p} \tilde{Q}_{m+2, m+p+2} - \tilde{P}_{m+2, m+p+2} \tilde{Q}_{m, m+p}] \tilde{Q}_{m+1, m+p+1} \\ &= [\tilde{P}_{m+1, m+p+1} \tilde{Q}_{m+2, m+p+2} - \tilde{P}_{m+2, m+p+2} \tilde{Q}_{m+1, m+p+1}] \tilde{Q}_{m, m+p} \end{aligned}$$

yields the equation

$$\begin{aligned} & (-1)^m \frac{\tilde{\Delta}_{m, m+p+1}}{\tilde{\Delta}_{m-1, m+p}} x^{2m+p+1} \tilde{Q}_{m+2, m+p+2} - (r'_{m, m+p} + s'_{m, m+p} x) x^{2m+p+1} \tilde{Q}_{m+1, m+p+1} \\ &= (-1)^m (\tilde{\Delta}_{m+1, m+p+2} / \tilde{\Delta}_{m, m+p+1}) x^{2m+p+3} \tilde{Q}_{m, m+p}. \end{aligned}$$

On dividing this equation by $(-1)^m (\tilde{\Delta}_{m, m+p+1} / \tilde{\Delta}_{m-1, m+p}) x^{2m+p+1}$ and transposing we get

$$\begin{aligned} \tilde{Q}_{m+2, m+p+2} &= (r''_{m, m+p} + s''_{m, m+p} x) \tilde{Q}_{m+1, m+p+1} \\ &+ (\tilde{\Delta}_{m-1, m+p} \tilde{\Delta}_{m+1, m+p+1} / \tilde{\Delta}_{m, m+p+1}^2) x^2 \tilde{Q}_{m, m+p}. \end{aligned}$$

Setting $x = 0$ and using the fact that $Q_{m,n}(0) = 1$ it follows that $r''_{m, m+p} = 1$. Then introducing the notation

$$\begin{aligned} s_{m, m+p} &= s''_{m, m+p}, \\ t_{m, m+p} &= -(\tilde{\Delta}_{m-1, m+p} \tilde{\Delta}_{m+1, m+p+1} / \tilde{\Delta}_{m, m+p+1}^2), \end{aligned}$$

one gets

$$(13) \quad \tilde{Q}_{m+2, m+p+2} = (1 + s_{m, m+p} x) \tilde{Q}_{m+1, m+p+1} - t_{m, m+p} x^2 \tilde{Q}_{m, m+p}.$$

The numbers $s_{m, m+p}$ and $t_{m, m+p}$ are real and in fact using (4) and (5) it is easy to see that $t_{m, m+p} > 0$. Now consider the new polynomials defined by

$$(14) \quad Q_{m, m+p}^*(x) = x^m \tilde{Q}_{m, m+p} \left(\frac{1}{x} \right).$$

From (13) and (14) it follows that

$$(15) \quad Q_{m+2, m+p+2}^* = (s_{m, m+p} + x) Q_{m+1, m+p+1}^* - t_{m, m+p} Q_{m, m+p}^*.$$

One now proceeds as in the theory of G -fractions; see [2, p. 204]. Let

$$(16) \quad Q_{m, m+p}^*(\bar{x}) Q_{m+1, m+p+1}^*(x) = X_{m, m+p} + i Y_{m, m+p}.$$

Multiplying both sides of (15) by $Q_{m+1, m+p+1}^*(\bar{x})$ and equating imaginary parts we get

$$Y_{m+1, m+p+1} = \eta |Q_{m+1, m+p+1}^*|^2 + t_{m, m+p} Y_{m, m+p},$$

where $x = \xi + i\eta$. Hence, if $\eta \neq 0$,

$$Y_{m+1, m+p+1}/\eta \geq t_{m, m+p}(Y_{m, m+p}/\eta),$$

and since $Y_{0, p} = \eta$ it follows that

$$(17) \quad \frac{Y_{m+1, m+p+1}}{\eta} \geq \prod_{i=0}^m t_{i, i+p} > 0.$$

From (16) and (17) we see that if $\eta \neq 0$, then $Q_{m+1, m+p+1}^*(x) \neq 0$, and hence the polynomial has only real zeroes. Since the roots of $\tilde{Q}_{m+1, m+p+1}(x)$ are the reciprocals of the roots of $Q_{m+1, m+p+1}^*$, the lemma follows.

LEMMA 3. *The polynomials $Q_{m, m+p}(x)$ have all their zeroes in the set S_θ , and, in fact, for any compact set G , bounded away from S_θ , there exists a $\delta > 0$ such that*

$$|Q_{m, m+p}(x)| \geq \delta^m \quad \text{for all } x \in G.$$

Proof. From (4)–(7) it follows easily that

$$\Delta_{m-1, m+p} = \exp(-m(m+p+1)i\theta) \tilde{\Delta}_{m-1, m+p},$$

and

$$\Delta_{m, m+p}(x) = \exp(-m(m+p+1)i\theta) \tilde{\Delta}_{m, m+p}(xe^{-i\theta}),$$

so that

$$Q_{m, m+p}(x) = \tilde{Q}_{m, m+p}(xe^{-i\theta}).$$

It is therefore sufficient to prove the lemma in the case $\theta = 0$. Hence we consider the function $\tilde{f}(x)$ introduced above and we assume that G is bounded away from S_0 .

For $0 < \delta_1 < |a_1|$, define

$$\begin{aligned} G_1(\delta_1) &= \{x: |\operatorname{Im}(x)| \geq \delta_1\}, \\ G_2(\delta_1) &= \{x: |\operatorname{Re}(x)| \leq |a_1| - \delta_1\}, \\ G_3(\delta_1) &= \{x: \operatorname{Re}(x) \leq |a_1| - \delta_1\}. \end{aligned}$$

Now since G is bounded away from S_0 , we may choose $\delta_1 > 0$ such that

$$(18) \quad \begin{aligned} G &\subseteq G_1(\delta_1) \cup G_3(\delta_1) \quad \text{in case A, and} \\ G &\subseteq G_1(\delta_1) \cup G_2(\delta_1) \quad \text{in cases B and C.} \end{aligned}$$

Since G is bounded, choose R such that $G \subseteq \{x: |x| \leq R\}$. We claim that $\delta = \delta_1/R$ satisfies the requirements of the lemma. To show this, let u_1, u_2, \dots, u_m denote the roots of $\tilde{Q}_{m,m+p}(x)$. From (14) we have

$$\begin{aligned} |\tilde{Q}_{m,m+p}(x)| &= |x|^m \left| Q_{m,m+p}^* \left(\frac{1}{x} \right) \right| = |x|^m \prod_{i=1}^m \left| \frac{1}{x} - \frac{1}{u_i} \right| \\ &\geq |x|^m \prod_{i=1}^m \left| \operatorname{Im} \left(\frac{1}{x} - \frac{1}{u_i} \right) \right| = |x|^m \prod_{i=1}^m \left| \operatorname{Im} \left(\frac{1}{x} \right) \right| \\ &= |x|^m \left(\frac{|\eta|}{|x|^2} \right)^m = \left| \frac{\eta}{x} \right|^m. \end{aligned}$$

Therefore for $x \in G_1(\delta_1) \cap G$ we have

$$(19) \quad |Q_{m,m+p}(x)| \geq |\eta/x|^m \geq (\delta_1/R)^m = \delta^m.$$

Furthermore, since $|1 - (x/u_i)| \geq |1 - (\xi/u_i)|$ we get

$$|\tilde{Q}_{m,m+p}(x)| = \prod_{i=1}^m |1 - (x/u_i)| \geq \prod_{i=1}^m |1 - (\xi/u_i)| = |\tilde{Q}_{m,m+p}(\xi)|.$$

Assume $x \in G_2(\delta_1)$. Then

$$\xi/a_i \leq |\xi/a_i| \leq |\xi/a_1| \leq |a_1| - \delta_1 / |a_1| = 1 - \delta_1 / |a_1|$$

so that

$$1 - \xi/a_i \geq \delta_1 / |a_1| \geq \delta_1 / R = \delta.$$

Hence for all $k \in S(m)$,

$$\prod_{\nu=1}^m (1 - \xi/a_{k_\nu}) \geq \delta^m.$$

It therefore follows from (5)–(7) and the fact that $T_{m,m+p}^k > 0$ that

$$(20) \quad |\tilde{Q}_{m,m+p}(x)| \geq |\tilde{Q}_{m,m+p}(\xi)| \geq \delta_m.$$

The lemma now follows in cases B and C by (18)–(20).

In case A, $a_i > 0$ for all i so that for $x \in G_3(\delta_1)$,

$$\xi/a_i \leq (a_1 - \delta_1)/a_1 = 1 - \delta_1/a_1,$$

and hence $1 - \xi/a_i \geq \delta_1/a_1 \geq \delta_1/R = \delta$. Consequently we get, as before, $|\tilde{Q}_{m,m+p}(x)| \geq |\tilde{Q}_{m,m+p}(\xi)| \geq \delta^m$. This together with (18) proves the lemma in case A as well.

Proof of the Theorem. Let G be as in the last lemma. Hence $G \subseteq \{x: |x| \leq R\}$ and there exists $\delta > 0$ such that $|Q_{i,i+p}(x)| \geq \delta^i$ for all $x \in G$ and for all i . Choose N such that

$$|a_N| > 4R/\delta,$$

and so

$$|2R/a_i \delta| < \frac{1}{2} \quad \text{for } i \geq N.$$

By Lemma 1,

$$\begin{aligned} & \left| (-1)^i \frac{A_{i,i+p+1}}{A_{i-1,i+p}} \frac{x^{2i+p+1}}{Q_{i,i+p} Q_{i+1,i+p+1}} \right| \\ & \leq |c_{p+1}| \left| \frac{a_1 a_2 \cdots a_i}{2 \cdot 2 \cdots 2} \right|^{-2} \frac{R^{2i+p+1}}{\delta^i \delta^{i+1}} = |c_{p+1}| \frac{R^{p+1}}{\delta} \left| \frac{2R}{a_1 \delta} \frac{2R}{a_2 \delta} \cdots \frac{2R}{a_i \delta} \right|^2 \\ & \leq |c_{p+1}| \frac{R^{p+1}}{\delta} \left| \frac{2R}{a_1 \delta} \cdots \frac{2R}{a_N \delta} \right|^2 \left(\frac{1}{2^{i-N}} \right)^2 \leq M 4^{-i}, \quad \text{for all } x \in G. \end{aligned}$$

It follows that the sum (10) converges uniformly to a holomorphic function $g(x)$ in G . Since we also have uniform convergence in a neighborhood of the origin and since $f_{m,m+p}^{(n)}(0) = n!c_n$ for $n < 2m+1$ we see that $g_{(0)}^{(n)} = n!c_n$. Hence $g(x) = f(x)$ for all $x \in G$. This completes the proof of the theorem.

EXAMPLES. Some simple examples to which our theorem applies are: $\tan x$, $\cot x - 1/x$, $[f''(x)/f'(x)] + (1/x)$, $\tanh x$ and $\coth x$.

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